100 Years

NORTH SYDNEY BOYS HIGH SCHOOL

1912 ~ 2012

MATHEMATICS (EXTENSION 1)

2012 HSC Course Assessment Task 4

August 15, 2012

General instructions

- Working time – 50 min.
- **Commence each new question on a new page**.
- Write using blue or black pen. Where diagrams are to be sketched, these may be done in pencil.
- Board approved calculators may be used.
- All necessary working should be shown in every question. Marks may be deducted for illegible or incomplete working.
- Attempt **all** questions.
- At the conclusion of the examination, bundle the booklets used in the correct order within this paper and hand to examination supervisors.

Class (please ✔)

- ◯ 12M3C – Ms Ziaziaris
- ◯ 12M3D – Mr Lowe
- ◯ 12M3E – Mr Lam
- ◯ 12M4A – Mr Lin
- ◯ 12M4B – Mr Ireland
- ◯ 12M4C – Mr Fletcher

STUDENT NUMBER .... **# BOOKLETS USED**: ....

Marker's use only.

QUESTION	1	2	3	4	5	**Total**	%
MARKS	/8	/7	/9	/7	/8	/39	

Question 1 (8 Marks) Commence a NEW page. **Marks**

(a) Prove $\dfrac{d}{dx}\left(\dfrac{1}{2}v^2\right) = \dfrac{d^2x}{dt^2}$. **2**

(b) The velocity of a particle moving along the x axis is given by

$$v^2 = 4\left(1+x^2\right)^2$$

Initially, the particle is at $x = 1$ with velocity $v = 4\,\text{ms}^{-1}$.

i. Find value of the initial acceleration. **2**

ii. Show the displacement of the particle as a function of time is **3**

$$x = \tan\left(2t + \frac{\pi}{4}\right)$$

iii. Explain why $0 \le t < \frac{\pi}{8}$ for this motion to be valid. **1**

Question 2 (7 Marks) Commence a NEW page. **Marks**

(a) Consider the expansion of $(1+2x)^n$.

i. Write down an expression for the coefficient of the term in x^4. **2**

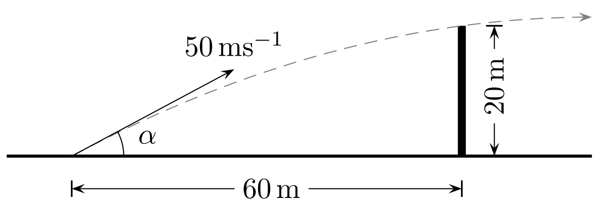
ii. The ratio of the coefficient of the term in x^4 to that of the term in x^6 is $5 : 8$. Find the value of n. **3**

(b) Find the coefficient of x^3 in the expansion of $(1-2x)^{18}(1+3x)^{17}$. **2**

Question 3 (9 Marks) Commence a NEW page. **Marks**

A particle is projected from ground level with an initial speed of $50\,\text{ms}^{-1}$ towards a wall which is 60 m from the point of projection, and 20 m high and just scrapes past the wall (i.e. does not collide with it).

Take $g = 10\,\text{ms}^{-2}$.



(a) Show that the equations of motion for this system are **3**

$$\begin{cases} x = 50t\cos\alpha \\ y = -5t^2 + 50t\sin\alpha \end{cases}$$

(b) Show that $9\tan^2\alpha - 75\tan\alpha + 34 = 0$. **3**

(c) Hence or otherwise, find the angle(s) of projection for which the particle will just scrape past the wall, correct to the nearest degree. **3**

Question 4 (7 Marks) Commence a NEW page. **Marks**

Samuel Sung opened a warehouse on 1 July 2012, distributing *SolarSystem S3* mobile phones to retailers.

His initial stock was 10 000 units of the phone. During any month, he sells 25% of the existing stock at the beginning of that month. In order to keep up with demand, he purchases an additional 100 phones on the last day of each month.

(a) Show that the number of phones in the warehouse at the end of the second month is **2**

$$A_2 = 10\,000 \times 0.75^2 + 100(1 + 0.75)$$

(b) Show that A_n, the number of *SolarSystem S3* phones in stock after n months is given by **3**

$$A_n = 9\,600 \times 0.75^n + 400$$

(c) After how many months of opening the warehouse will Samuel distribute less than 500 phones to retailers? **2**

Question 5 (8 Marks) Commence a NEW page. **Marks**

(a) By considering the expansion of $(1+x)^n$, show that **3**

$$2\binom{n}{2} + 6\binom{n}{3} + 12\binom{n}{4} + \cdots + n(n-1)\binom{n}{n} = n(n-1)2^{n-2}$$

(b) i. State the number of terms in this geometric series: **1**

$$1 + (1+x) + (1+x)^2 + (1+x)^3 + \cdots + (1+x)^n$$

ii. Express $1 + (1+x) + (1+x)^2 + (1+x)^3 + \cdots + (1+x)^n$ in simplest terms, using the formula for the sum of a geometric progression. **2**

iii. By considering the coefficient of x^r, where $0 < r \leq n$, in the expansion of **2**

$$1 + (1+x) + (1+x)^2 + (1+x)^3 + \cdots + (1+x)^n$$

Prove that

$$\binom{n}{r} + \binom{n-1}{r} + \binom{n-2}{r} + \cdots + \binom{r}{r} = \binom{n+1}{r+1}$$

End of paper.

STANDARD INTEGRALS

$$\int x^n \, dx = \frac{1}{n+1} x^{n+1} + C, \qquad n \neq -1; \quad x \neq 0 \text{ if } n < 0$$

$$\int \frac{1}{x} \, dx = \ln x + C, \qquad x > 0$$

$$\int e^{ax} \, dx = \frac{1}{a} e^{ax} + C, \qquad a \neq 0$$

$$\int \cos ax \, dx = \frac{1}{a} \sin ax + C, \qquad a \neq 0$$

$$\int \sin ax \, dx = -\frac{1}{a} \cos ax + C, \qquad a \neq 0$$

$$\int \sec^2 ax \, dx = \frac{1}{a} \tan ax + C, \qquad a \neq 0$$

$$\int \sec ax \tan ax \, dx = \frac{1}{a} \sec ax + C, \qquad a \neq 0$$

$$\int \frac{1}{a^2 + x^2} \, dx = \frac{1}{a} \tan^{-1} \frac{x}{a} + C, \qquad a \neq 0$$

$$\int \frac{1}{\sqrt{a^2 - x^2}} \, dx = \sin^{-1} \frac{x}{a} + C, \qquad a > 0, -a < x < a$$

$$\int \frac{1}{\sqrt{x^2 - a^2}} \, dx = \ln \left(x + \sqrt{x^2 - a^2} \right) + C, \qquad x > a > 0$$

$$\int \frac{1}{\sqrt{x^2 + a^2}} \, dx = \ln \left(x + \sqrt{x^2 + a^2} \right) + C$$

NOTE: $\ln x = \log_e x, \; x > 0$

Suggested Solutions

Question 1 (Ziaziaris)

(a) (2 marks)

$$\begin{aligned}\frac{d}{dx}\left(\frac{1}{2}v^2\right) &= \frac{d}{dv}\left(\frac{1}{2}v^2\right)\frac{dv}{dx} \quad \text{(chain rule)}\\ &= v\frac{dv}{dx}\\ &= \frac{dx}{dt}\cdot\frac{dv}{dx}\\ &= \frac{dv}{dt} = \frac{d^2x}{dt^2}\\ &= a\end{aligned}$$

(b) i. (2 marks)

✓ [1] for differentiating to obtain

$$a = 8x(1+x^2)$$

✓ [1] obtaining $a = 8\,\text{ms}^{-2}$ after substitution and evaluation.

$$\begin{aligned}v^2 &= 4\left(1+x^2\right)^2\\ \frac{d}{dx}\left(\frac{1}{2}v^2\right) &= \frac{d}{dx}\left(2\left(1+x^2\right)^2\right)\\ &= 2\cdot 2x\cdot 2\cdot(1+x^2)\\ &= 8x\left(1+x^2\right)\Big|_{x=1}\\ &= 16\,\text{ms}^{-2}\end{aligned}$$

ii. (3 marks)

✓ [1] for obtaining $\frac{dx}{dt} = 2(1+x^2)$.

✓ [1] for obtaining $\tan^{-1}x = 2t + C$.

✓ [1] for value of C.

$$v^2 = 4\left(1+x^2\right)^2$$

When $x = 1$, $v > 0$. Hence use positive square root.

$$\begin{aligned}v = \frac{dx}{dt} &= 2\left(1+x^2\right)\\ \frac{dx}{1+x^2} &= 2\,dt\end{aligned}$$

Integrating,

$$\tan^{-1}x = 2t + C$$

When $t = 0$, $x = 1$.

$$\begin{aligned}\tan^{-1}1 = \frac{\pi}{4} &= 0 + C\\ \therefore C &= \frac{\pi}{4}\\ \therefore x &= \tan\left(2t + \frac{\pi}{4}\right)\end{aligned}$$

iii. (1 mark)

✓ [1] awarded only for fully justified answers.

As it is not physically possible to cross $t = \frac{\pi}{2}$ when $x = \tan t$ (which makes particle "teleport" from $x = \infty$ to $x = -\infty$!), inspect the domain of $\tan t$ in its first period only:

$$D = \left\{t : -\frac{\pi}{2} < t < \frac{\pi}{2}\right\}$$

Inspecting the domain of $\tan\left(2t + \frac{\pi}{4}\right)$:

$$\begin{aligned}D &= \left\{t : -\frac{\pi}{2} < 2t + \frac{\pi}{4} < \frac{\pi}{2}\right\}\\ &= \left\{t : -\frac{3\pi}{4} < 2t < \frac{\pi}{4}\right\}\end{aligned}$$

As $t > 0$,

$$D = \left\{t : 0 \le 2t < \frac{\pi}{4}\right\} = \left\{t : 0 \le t < \frac{\pi}{8}\right\}$$

Question 2 (Lowe)

(a) i. (2 marks)

✓ [−1] if the term (rather than coefficient only) is given.

$$(1+2x)^n = \sum_{k=0}^{n}\binom{n}{k}2^k x^k$$

Coefficient of term in x^4:

$$\binom{n}{4}2^4$$

ii. (3 marks)

- ✓ [1] for simplifying ratio of coefficient of term in x^4 to term in x^6.
- ✓ [1] for obtaining quadratic after equating with ratio of $\frac{5}{8}$.
- ✓ [1] for justifying $n = 8$ only.

The term in x^6 is

$$\binom{n}{6}2^6$$

Ratio of term in x^4 to that in x^6:

$$\begin{aligned}\frac{\binom{n}{4}\times 2^4}{\binom{n}{6}\times 2^6} &= \frac{\dfrac{\cancel{n!}}{\cancel{4!}(n-4)(n-5)\cancel{(n-6)!}}\cancel{2^4}}{\dfrac{\cancel{n!}}{6\times 5\times \cancel{4!}\cancel{(n-6)!}}\cancel{2^6}\,2^2}\\ &= \frac{30}{4(n-4)(n-5)}\\ &= \frac{15}{2(n-4)(n-5)}\end{aligned}$$

The terms are in the ratio $5:8$:

$$\begin{aligned}\frac{\cancel{15}^{\,3}}{\cancel{2}(n-4)(n-5)} &= \frac{\cancel{5}}{\cancel{8}_4}\\ 12 &= (n-4)(n-5)\\ n^2-9n+20 &= 12\\ n^2-9n+8 &= 0\\ (n-8)(n-1) &= 0\\ \therefore n &= 8 \text{ only}\end{aligned}$$

as $n = 1$ produces no x^4 or x^6 term.

(b) (2 marks)

- ✓ [1] for pairs of k and r that add to required index.
- ✓ [1] for final answer.

$$\begin{aligned}&(1-2x)^{18}(1+3x)^{17}\\ &= \left(\sum_{k=0}^{18}\binom{18}{k}(-1)^k 2^k x^k\right)\left(\sum_{r=0}^{17}\binom{17}{r}3^r x^r\right)\end{aligned}$$

The typical term in this expansion is

$$\binom{18}{k}\binom{17}{r}(-1)^k 2^k 3^r x^{k+r}$$

The coefficient of x^3 appears when $k+r=3$:

k	r	$k+r$	Coefficient
0	3	3	$\binom{18}{0}\binom{17}{3}2^0 3^3$
1	2	3	$-\binom{18}{1}\binom{17}{2}2^1 3^2$
2	1	3	$\binom{18}{2}\binom{17}{1}2^2 3^1$
3	0	3	$-\binom{18}{3}\binom{17}{0}2^3 3^0$

Hence the coefficient of the term in x^3 is

$$\binom{18}{0}\binom{17}{3}2^0 3^3 - \binom{18}{1}\binom{17}{2}2^1 3^2 + \binom{18}{2}\binom{17}{1}2^2 3^1 - \binom{18}{3}\binom{17}{0}2^3 3^0$$

(Stop at this point and ignore any further working to simplify)

Question 3 (Fletcher)

(a) (3 marks)

- ✓ [1] for correct basis equations

$$\ddot{x} = 0 \qquad \ddot{y} = -g$$

- ✓ [1] for each correct derivation of x and y in terms of t.

$$\begin{cases}\ddot{x} = 0\\ \ddot{y} = -g\end{cases}$$

Finding x by integrating twice:

$$\dot{x} = \int \ddot{x}\,dt = \int 0\,dt = C_1$$

When $t = 0$, the component of velocity is

$$\begin{aligned}\dot{x} &= 50\cos\alpha\\ x &= \int \dot{x}\,dt = \int 50\cos\alpha\,dt\\ &= 50t\cos\alpha + C_2\end{aligned}$$

When $t = 0$, $x = 0$. Hence $C_2 = 0$.

$$\therefore x = 50t\cos\alpha$$

Finding y by integrating twice:

$$\dot{y} = \int \ddot{y}\,dt = \int -g\,dt = -gt + C_3$$

When $t = 0$, the component of velocity is

$$\begin{aligned} \dot{y} &= -g(0) + C_3 = 50\sin\alpha \\ \therefore C_3 &= 50\sin\alpha \\ \therefore \dot{y} &= -gt + 50\sin\alpha \\ y &= \int \dot{y}\,dt \\ &= \int -gt + 50\sin\alpha\,dt \\ &= -\frac{1}{2}gt^2 + 50t\sin\alpha + C_4 \end{aligned}$$

When $t = 0$, $y = 0$. Hence $C_4 = 0$.

$$\therefore y = -\frac{1}{2}gt^2 + 50t\sin\alpha$$

(b) (3 marks)

- ✓ [1] for Cartesian equation, eliminating t.
- ✓ [1] for substituting $x = 60$, $y = 20$.
- ✓ [1] for final answer.

$$\begin{cases} y = -5t^2 + 50t\sin\alpha \\ x = 50t\cos\alpha \end{cases}$$

$$\therefore t = \frac{x}{50\cos\alpha}$$

Substitute to $y = -5t^2\cdots$:

$$\begin{aligned} y &= -5\left(\frac{x}{50\cos\alpha}\right)^2 + 50\left(\frac{x}{50\cos\alpha}\right)\sin\alpha \\ &= -\frac{\cancel{5}x^2}{\cancel{5}\times 5\times 10^2\cos^2\alpha} + x\tan\alpha \\ &= -\frac{x^2}{500}\sec^2\alpha + x\tan\alpha \\ &= -\frac{x^2}{500}\left(1+\tan^2\alpha\right) + x\tan\alpha \end{aligned}$$

When particle scrapes past, $x = 60$ and $y = 20$.

$$\underbrace{\cancelto{5}{20}}_{\times 5} = \underbrace{-\frac{\cancelto{9}{6\times 6\times 10^2}}{5\times\cancel{10^2}}\left(1+\tan^2\alpha\right) + \cancelto{15}{60}\tan\alpha}_{\times 5}$$

$$\begin{aligned} 25 &= -9 - 9\tan^2\alpha + 75\tan\alpha \\ \therefore 9\tan^2\alpha - 75\tan\alpha + 34 &= 0 \end{aligned}$$

(c) (3 marks)

- ✓ [1] for $\tan\alpha = \frac{25\pm\sqrt{489}}{6}$ (or unsimplified equivalent)
- ✓ [1] each for $\alpha = 26^\circ$ or $\alpha = 83^\circ$

Let $m = \tan\alpha$.

$$\begin{aligned} 9m^2 - 75m + 34 &= 0 \\ m &= \frac{75\pm\sqrt{75^2 - 4(9)(34)}}{2\times 9} \\ &= \frac{75\pm 3\sqrt{489}}{18} \\ &= \frac{25\pm\sqrt{489}}{18} \\ \therefore \tan\alpha &= \frac{25\pm\sqrt{489}}{18} \\ \alpha &= 26^\circ, 83^\circ \end{aligned}$$

Question 4 (Lin)

(a) (2 marks)

- ✓ [1] for A_1.
- ✓ [1] for A_2.

$$\begin{aligned} A_1 &= 10\,000\times 0.75 + 100 \\ A_2 &= A_1\times 0.75 + 100 \\ &= (10\,000\times 0.75 + 100)\times 0.75 \\ &\quad + 100 \\ &= 10\,000\times 0.75^2 + 100(1 + 0.75) \end{aligned}$$

(b) (3 marks)

- ✓ [1] for correctly generalising to A_n.
- ✓ [1] for $S_n = 4(1 - 0.75^n)$.
- ✓ [1] for final answer.

$$\begin{aligned} A_n &= 10\,000\times 0.75^n \\ &\quad + 100\underbrace{\left(1 + 0.75 + \cdots + 0.75^{n-1}\right)}_{\text{GP:}a=1, r=0.75} \end{aligned}$$

$$\left| \begin{aligned} S_n &= \frac{1\left(1 - 0.75^n\right)}{1 - 0.75} = \frac{\left(1 - 0.75^n\right)}{\frac{1}{4}} \\ &= 4\left(1 - 0.75^n\right) \end{aligned} \right.$$

$$\begin{aligned} \therefore A_n &= 10\,000\times 0.75^n + 400\left(1 - 0.75^n\right) \\ &= 0.75^n\left(10\,000 - 400\right) + 400 \\ &= 9\,600\times 0.75^n + 400 \end{aligned}$$

(c) (2 marks)

✓ [1] for $A_n \times 0.25 < 500$.

✓ [1] for final answer.

- The number of phones distributed is $A_n \times 0.25$ (distributes 25% of what is in the warehouse).

$$\begin{aligned}(9\,600 \times 0.75^n + 400) \times 0.25 &< 500\\ 9\,600 \times 0.75^n + 400 &< \frac{500}{0.25} = 2000\\ \underset{\div 9\,600}{9\,600} \times 0.75^n &< \underset{\div 9\,600}{1\,600}\\ 0.75^n &< \frac{1}{6}\\ n \log 0.75 &< \log \frac{1}{6}\\ n &> \frac{\log \frac{1}{6}}{\log 0.75} = 6.228\cdots\end{aligned}$$

Less than 500 phones will be distributed after 7 months.

Question 5 (Ireland)

(a) (3 marks)

✓ [1] for correctly differentiating each time.

✓ [1] for substituting $x = 1$ and obtaining correct expression.

$$(1+x)^n = \sum_{k=0}^{n} \binom{n}{k} x^k$$

Differentiating,

$$n(1+x)^{n-1} = \binom{n}{1} + 2\binom{n}{2}x + \cdots + n\binom{n}{n}x^{n-1}$$

Differentiate again,

$$\begin{aligned}&n(n-1)(1+x)^{n-2}\\ &= 2\binom{n}{2} + 3\cdot 2\binom{n}{3}x + \cdots + n(n-1)\binom{n}{n}x^{n-2}\end{aligned}$$

When $x = 1$,

$$n(n-1)2^{n-2} = 2\binom{n}{2} + 6\binom{n}{3} + \cdots + n(n-1)\binom{n}{n}$$

(b) i. (1 mark) $n+1$.

ii. (2 marks)

✓ [1] for correct substitution into sum of GP formula.

✓ [1] for final answer.

$$1 + (1+x) + (1+x)^2 + (1+x)^3 + \cdots + (1+x)^n$$

GP: $a = 1$, $r = (1+x)$.

$$\begin{aligned}S_{n+1} &= \frac{1\left((1+x)^{n+1} - 1\right)}{(1+x) - 1}\\ &= \frac{1}{x}\left((1+x)^{n+1} - 1\right)\end{aligned}$$

iii. (2 marks)

✓ [1] for finding coef in x^r as $\binom{n}{r} + \binom{n-1}{r} + \cdots + \binom{r}{r}$.

✓ [1] for obtaining coef of x^{r+1} which is $\binom{n+1}{r+1}$.

$$\begin{aligned}&\frac{1}{x}\left((1+x)^{n+1} - 1\right)\\ &= 1+(1+x)+(1+x)^2+\cdots+(1+x)^n\end{aligned}$$

Examine the coefficient of x^r in $\frac{1}{x}\left((1+x)^{n+1} - 1\right)$:

- x^r term appears when the power is $\geq r$. Relevant terms are:

$$\binom{r}{r} \quad \binom{r+1}{r} \quad \binom{r+2}{r} \cdots \binom{n}{r}$$

In $1+(1+x)+(1+x)^2+\cdots+(1+x)^n$, the term required is x^{r+1} as x^{-1} exists to reduce the index by 1. The coefficient is thus

$$\binom{n+1}{r+1}$$

Equating coefficients in x^r,

$$\begin{aligned}&\binom{r}{r} + \binom{r+1}{r} + \cdots + \binom{n-1}{r} + \binom{n}{r}\\ &\qquad = \binom{n+r}{r+1}\end{aligned}$$